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Scattering properties of point dipole interactions

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Abstract
The scattering properties of a three-parameter family of point dipole-like interactions constructed from a sequence of barrier-well rectangles are studied in the zero-range limit. Besides the real (unrenormalized) δ'-interaction, the derivative of Dirac's delta function, a whole family of point dipole interactions with a renormalized coupling constant are analysed. Depending on the parameter values, all these interactions being self-adjoint extensions of the one-dimensional Schrödinger operator are shown to be divided into four types: (i) interactions will full transparency, (ii) non-transparent interactions, (iii) partially transparent interactions acting effectively as a δ-interaction and (iv) interactions with partial transparency at discrete resonant values of the coupling constant.

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1. Introduction

Point and contact interactions are widely used in various areas of quantum physics (see [1, 2] and references therein including a large number of other applications, e.g., [3–8]). Intuitively, these interactions are understood as sharply localized potentials, exhibiting a number of interesting and intriguing features. Applications of these models to condensed matter physics (see, e.g., [9–13]) are of particular interest nowadays, mainly because of the rapid progress in fabricating nanoscale quantum devices. Other applications arise in optics, for instance, in dielectric media where electromagnetic waves scatter at boundaries or thin layers [14].

In the following we use the quantum-mechanical terminology and consider the limit which neglects the interactions between electrons. In this case the one-dimensional Schrödinger equation with a potential $V(x)$ for a stationary state reads

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

where the prime stands for the differentiation with respect to the spatial coordinate x, $\psi(x)$ is the wavefunction for a particle of mass m (we use units in which $\hbar^2/2m = 1$) and E is (positive, zero or negative) energy. The present paper deals with the scattering properties of equation (1) where the potential $V(x)$ is a point interaction defined by the following three-parameter boundary conditions [2, 15–18]:

$$\psi(+0) - \alpha\psi(-0) = \beta\psi'(-0), \qquad \psi'(+0) - \rho\psi'(-0) = \gamma\psi(-0) \tag{2}$$

with α, β, γ and ρ being real constants subject to the constraint

$$\alpha\rho - \beta\gamma = 1. \tag{3}$$

Due to condition (3), among these four constants three are independent. If we suppose furthermore that the interaction $V(x)$ is invariant under space reflection $x \to -x$, these boundary conditions become invariant under the transformation

$$\psi(\pm 0) \to \psi(\mp 0) \qquad \text{and} \qquad \psi'(\pm 0) \to -\psi'(\mp 0) \tag{4}$$

if and only if $\alpha = \rho$.

Thus, in the particular case of the point potential in the form of Dirac's delta function,

$$V(x) = g\delta(x) \tag{5}$$

with g being a coupling constant, the constant values in (2) become

$$\alpha = \rho = 1, \qquad \beta = 0, \qquad \gamma = g. \tag{6}$$

This type of boundary conditions (2) assumes that the wavefunction $\psi(x)$ is continuous but its first derivative is discontinuous at the singularity point ($x = 0$). This is a quite simple example of point interactions in one dimension. Because of the continuity of the wavefunction $\psi(x)$, the product $\delta(x)\psi(x)$ becomes well-defined at $x = 0$ and therefore the solution of equation (1) with potential (5) is unique.

Another choice of the constants in boundary conditions (2), namely

$$\alpha = \rho = 1, \qquad \beta = \lambda, \qquad \gamma = 0, \tag{7}$$

defines the so-called δ'-interaction [2, 11, 15, 16], assuming a discontinuity of wavefunctions but their continuous derivatives at the singularity point. Note since $\alpha = \rho$, the δ'-interaction so defined is invariant under the transformation $x \to -x$. This is in contrast to the potential $V(x)$ defined in the standard way as the derivative of Dirac's delta function:

$$V(x) = \lambda\delta'(x), \qquad \delta'(x) \doteq \mathrm{d}\delta(x)/\mathrm{d}x. \tag{8}$$

Therefore, as pointed out by Exner [11] and Coutinho *et al* [18], the definition of the δ'-interaction through boundary conditions (2) with values (7) has little resemblance to what the name δ' suggests.

In the following we are dealing only with the case of distribution (8) defined, e.g., on the space of test functions formed from all infinitely differentiable functions defined on the whole axis $-\infty < x < \infty$, which tend to zero as $|x| \to \infty$, together with their derivatives of all orders, more rapidly than any power of $1/|x|$. We refer to potential (8) as a point *dipole* interaction [15]. Due to discontinuity of the wavefunction $\psi(x)$, the product $\delta'(x)\psi(x)$ is not well-defined at $x = 0$. Therefore Šeba [15] suggested considering a renormalized point (dipole-like and odd) interaction defined as the limit

$$V(x) \doteq \lim_{\varepsilon\to 0}(\lambda/2\varepsilon^{\tau})[\delta(x+\varepsilon) - \delta(x-\varepsilon)], \tau > 0, \tag{9}$$

as $\varepsilon \to 0$. In the particular case $\tau = 1$ limit (9) gives unrenormalized interaction (8). In a rigorous way Šeba [15] has proved that for $\tau < 1/2$ limit (9) is trivial, i.e., the system behaves as if the zero-range potential is absent, while for $\tau > 1/2$ the system splits into two independent subsystems separated on the half-axes $-\infty < x < 0$ and $0 < x < \infty$. The only interesting case is $\tau = 1/2$ for which there exists a non-trivial interaction between the subsystems. In physical terms this means that the δ'-system becomes fully transparent for $\tau < 1/2$, fully non-transparent for $\tau > 1/2$ and partially transparent if $\tau = 1/2$. Moreover, it appears that in the latter case the system behaves as a δ-interaction [15].

Recently [19] it has been suggested to regularize the distribution $\delta'(x)$ in (8) by a sequence of rectangular stepwise functions chosen in the form of a barrier and a well with a squeezing parameter. As a result, it has been shown that there exists a discrete set of coupling constant values $\{\lambda_n\}_{n=1}^{\infty}$ at which interaction (8) is partially transparent. These values satisfy the transcendental equation

$$\tan \sqrt{\lambda} = \tanh \sqrt{\lambda}. \tag{10}$$

For other values of λ the transmission appears to be zero. In this regard the reasonable question arises: whether these resonance values are characteristic for the singular potential $\lambda\delta'(x)$ or they depend on the way in which the derivative of Dirac's delta function is obtained in the weak limit. Therefore, the main purpose of the present paper is to study this problem. As a result, we find the interesting feature of the δ'-interaction: the resonances in the λ-space are *not* specific for a given point interaction $\lambda\delta'(x)$, they depend on the way in which the distribution $\delta'(x)$ is regularized and at least a one-parameter discrete subset in the λ-space can be constructed by using an asymmetry between the barrier and well rectangles. In addition, it is reasonable to generalize Šeba's results [15] using asymmetric barrier-well rectangles for regularization of interaction (8). Note also that the derivative of Dirac's delta function is only a mathematical idealization and in practice finite structures, preferably rectangle-like layers, are most often used. In particular, the point dipole may be used as a limiting case of a situation in which a small region of large repulsive potential is immediately followed by a small region of large attractive potential. The analytical results obtained in this paper, which describe in detail the transmission properties of squeezed barrier-well sequences of rectangular shape that approach $\delta'(x)$, can be used in particular for experimental realization and production of nano-devices with sharp resonances to be described below.

2. A rectangular regularization of the potential $\lambda\delta'(x)$

In the following we consider a general renormalized case of the potential $\lambda\delta'(x)$. To this end we approximate it by two rectangles, a barrier of height h and width l immediately followed by a well of depth d and width r. Hence the regularized potential is assumed to be a stepwise function defined by

$$V_{lr}(x) \doteq \lambda \begin{cases} 0 & \text{for } -\infty < x < -l \\ h & \text{for } -l < x < 0 \\ -d & \text{for } 0 < x < r \\ 0 & \text{for } r < x < \infty \end{cases} \tag{11}$$

with four positive parameters h, l, d and r.

We are looking for the positive-energy solution of equation (1) with potential (11) given in the form

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R\,\mathrm{e}^{-\mathrm{i}kx} & \text{for } -\infty < x < -l \\ A_1\,\mathrm{e}^{px} + B_1\,\mathrm{e}^{-px} & \text{for } -l < x < 0 \\ A_2 \sin(qx) + B_2 \cos(qx) & \text{for } 0 < x < r \\ T\,\mathrm{e}^{\mathrm{i}kx} & \text{for } r < x < \infty \end{cases} \tag{12}$$

where R and T are reflection and transmission coefficients (from the left), respectively, and

$$k = \sqrt{E},\ p = \sqrt{\lambda h - E} \qquad \text{and} \qquad q = \sqrt{\lambda d + E}. \tag{13}$$

The unknown coefficients A_j and B_j, $j = 1, 2$, are eliminated in a standard way by matching the solutions at the boundaries $x = -l, 0, r$. As a result, the reflection and transmission coefficients can be written in the following form:

$$R = -\frac{u + \mathrm{i}v}{\Delta}\,\mathrm{e}^{-2\mathrm{i}kl} \qquad \text{and} \qquad T = \frac{2}{\Delta}\,\mathrm{e}^{-\mathrm{i}k(l+r)} \tag{14}$$

where

$$u = \frac{\lambda(h + d)}{pq}\sinh(pl)\sin(qr), \tag{15}$$

$$v = \frac{\lambda}{k}\left[\frac{h}{p}\sinh(pl)\cos(qr) - \frac{d}{q}\cosh(pl)\sin(qr)\right], \tag{16}$$

with

$$\Delta = \left[\left(\frac{p}{q} - \frac{q}{p}\right)\sinh(pl) - \mathrm{i}\left(\frac{q}{k} + \frac{k}{q}\right)\cosh(pl)\right]\sin(qr) + \left[2\cosh(pl) + \mathrm{i}\left(\frac{p}{k} - \frac{k}{p}\right)\sinh(pl)\right]\cos(qr). \tag{17}$$

These finite-range expressions with arbitrary positive constants l and r will be used below in the zero-range limit accomplished in such a way that the barrier height h and the well depth d tend to infinity as $l \to 0$ and $r \to 0$. In this way we are able to consider a whole family of renormalized versions of point interaction (8).

3. Scattering coefficients in the zero-range limit

A point δ'-like interaction can be obtained from squeezing (as $l, r \to 0$) the barrier-well system (11), when both the barrier height and the well depth increase to infinity. Consider the general case

$$h = al^{-\mu} \qquad \text{and} \qquad d = bl^{-\nu} \tag{18}$$

with arbitrary positive constants a, b, μ and ν, where width l serves as a squeezing parameter. As regards the well width r, using the condition that for a δ'-like approximating sequence the area above the x-axis and the area below this axis must be equal, we obtain the relation between l and r:

$$r = \eta l^{1-\mu+\nu}, \qquad \eta \doteq a/b. \tag{19}$$

This relation shows how the behaviour of r depends on the squeezing of l. In general, we are interested in the case of a δ'-like point interaction when the distance r goes to zero as $l \to 0$. In this case, as follows from relation (19), the inequality

$$1 - \mu + \nu > 0 \tag{20}$$

has to be imposed as a necessary condition for obtaining point interactions in the zero-range limit. The set of all point interactions being a subset of the quadrant $\{\mu > 0, \nu > 0\}$ is shown in figure 1 by

$$\Omega_p = \{\mu > 0, \nu > 0 \,|\, \mu - 1 < \nu < \infty\}. \tag{21}$$

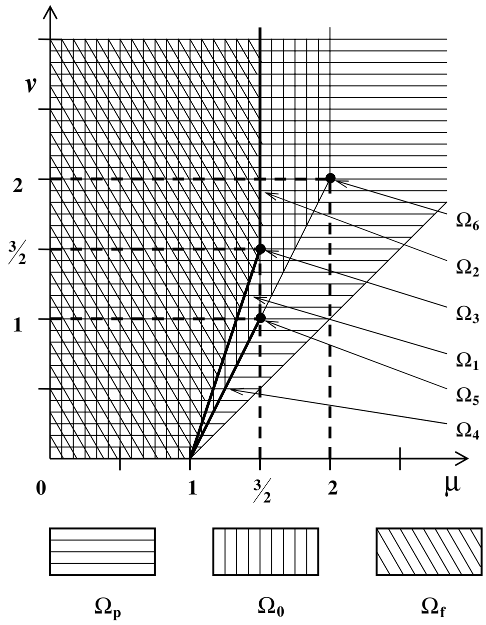

Figure 1. Diagram of existence of δ'-interactions. Non-trivial point interactions are located on lines $L_1 \doteq \Omega_1 \cup \Omega_2 \cup \Omega_3$, $L_2 \doteq \Omega_4 \cup \Omega_5$ and isolated point Ω_6.

Using now relations (13), (18) and (19), in the limit $l \to 0$ we find the following asymptotical behaviour:

$$p \to \sigma l^{-\mu/2}, \qquad q \to \eta^{-1/2}\sigma l^{-\nu/2} \qquad \text{and} \qquad qr \to \eta^{1/2}\sigma l^{1-\mu+\nu/2} \tag{22}$$

where

$$\sigma \doteq \sqrt{\lambda a}. \tag{23}$$

It follows immediately from these asymptotics that $pl \to 0$ and $qr \to 0$ simultaneously for the open set

$$\Omega_0 = \{\mu > 0, \nu > 0 \,|\, \mu < 2, 2(\mu - 1) < \nu < \infty\} \tag{24}$$

being a subset of Ω_p (see figure 1).

Using further expression (15) for u and asymptotics (22), we find the asymptotical behaviour

$$u \to \sigma^2 (l^{2-\mu} + \eta l^{2-2\mu+\nu}) \tag{25}$$

as $l \to 0$. Therefore everywhere in the region Ω_0 we have $u \to 0$. Using further equation (16) for v and taking into account the limits $pl \to 0$ and $qr \to 0$ including asymptotics (22), we obtain the following asymptotical behaviour of v as $l \to 0$:

$$v \to -\frac{\sigma^4}{3k}(l^{3-2\mu} + \eta l^{3-3\mu+\nu}). \tag{26}$$

As follows from these asymptotics, $v \to 0$ if both the inequalities $3 - 2\mu > 0$ and $3(1-\mu) + \nu > 0$ hold simultaneously. Therefore due to limits (25) and (26) the set

$$\Omega_f = \{\mu > 0, \nu > 0 \,|\, \mu < 3/2, 3(\mu - 1) < \nu < \infty\}, \tag{27}$$

being a subset of Ω_0 (see figure 1), appears to be a region of full transparency for the family of δ'-like point interactions defined through relations (18).

Let us now consider the boundary of the set Ω_f consisting of the two lines $\Omega_1 = \{\mu, \nu | 1 < \mu < 3/2, \nu = 3(\mu - 1)\}$ and $\Omega_2 = \{\mu, \nu | \mu = 3/2, 3/2 < \nu < \infty\}$ and the single point $\Omega_3 = \{\mu = \nu = 3/2\}$ connecting these lines. As shown above on this boundary we have $u \to 0$. As concerns other quantities, using asymptotics (26), we obtain finite values for v as well as for Δ. As a result, according to equations (14)–(17), we obtain the same scattering quantities as for the δ-interaction (5), i.e.,

$$R = \frac{1}{2\mathrm{i}k/g - 1} \qquad \text{and} \qquad T = \frac{1}{1 + \mathrm{i}g/2k} \tag{28}$$

with

$$g = -\frac{\sigma^4}{3} \begin{cases} \eta & \text{for } \Omega_1 \\ 1 & \text{for } \Omega_2 \\ 1+\eta & \text{for } \Omega_3. \end{cases} \tag{29}$$

Thus, for all these three sets, i.e., on the line $L_1 \doteq \Omega_1 \cup \Omega_2 \cup \Omega_3$, the transmission is the same as for δ-well potential (5) where the coupling constant g is given by equations (29).

A comparison with Šeba's renormalization [15] of δ'-interaction (9) can be carried out only for the antisymmetric (odd) system when $\eta = 1$ and $\mu = \nu$. Thus, for $\tau < 1/2$ the bisector $\mu = \nu$ lies inside the set Ω_f with full transparency, while for $\tau > 1/2$ this bisector belongs to the set where both the subsystems are separated. Therefore in (18) and (19) we have to choose

$$h = d = \frac{1}{2\varepsilon^\tau l}, \qquad r = l \quad \text{and} \quad \varepsilon = l/2. \tag{30}$$

Then in the case with $\tau = 1/2$ this choice corresponds to the point $\mu = \nu = 3/2$. According to (18) and (23) we obtain $a = b = 2^{-1/2}$ and therefore $\sigma^2 = \lambda/\sqrt{2}$. Thus, from (29) we get $g = -\lambda^2/3$, while the direct calculation of R and T for potential (9) at $\tau = 1/2$ in the limit $\varepsilon \to 0$ gives the same scattering quantities (28) but with $g = -\lambda^2/2$ [15, 18]. This difference is due to the different way of obtaining the point δ'-interaction. The former limit was performed in the order $l \to 0$ and then $\varepsilon \to 0$, while the latter one in the inverse order: $\varepsilon \to l/2$ and then $l \to 0$. In other words, we have obtained that on the line L_1, even the resulting point interaction is a Dirac delta function, the scattering quantities differ, i.e.,

$$\lim_{\varepsilon\to 0}\lim_{l\to 0}\{R, T\} \neq \lim_{l\to 0}\lim_{\varepsilon\to l/2}\{R, T\}. \tag{31}$$

A similar situation takes place for the next point on the bisector, namely $\mu = \nu = 2$, where the left-hand repeated limit in (31) leads to total separation of both the subsystems lying on the half-axes $-\infty < x < 0$ and $0 < x < \infty$, while the opposite sequence of limits, i.e., the right-hand repeated limit in (31), gives a non-trivial interaction between these subsystems as shown previously in [19].

Now we consider the case when $pl \to 0$ but qr tends to a non-zero constant as $l \to 0$. This non-zero limit occurs on the line $\nu = 2(\mu - 1)$. More precisely, according to asymptotics (22), on the line $L_2 \doteq \Omega_4 \cup \Omega_5$ where (see figure 1) $\Omega_4 \doteq \{\mu, \nu | 1 < \mu < 3/2, \nu = 2(\mu - 1)\}$ and $\Omega_5 \doteq \{\mu = 3/2, \nu = 1\}$ we have the limits $pl \to 0$ and $qr \to \sqrt{\eta}\sigma$. As a result, on this line one obtains

$$u \to \sqrt{\eta}\sigma \sin(\sqrt{\eta}\sigma), \tag{32}$$

while $v \to \infty$ and $\Delta \to \infty$, except for those values of the parameter σ which satisfy the equation

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\sigma. \tag{33}$$

The solution of this transcendental equation with respect to σ (or the coupling constant λ) represents a discrete set of positive values which we denote by $\{\bar{\sigma}_n\}_{n=1}^{\infty}$. Each $\bar{\sigma}_n$ can be found in the interval

$$n\pi\eta^{-1/2} < \bar{\sigma}_n < (n+1/2)\pi\eta^{-1/2}, \qquad n = 1, 2, \ldots. \tag{34}$$

At the values $\bar{\sigma}_n$'s on the line Ω_4, besides asymptotics (32), we also obtain the limits

$$v \to 0 \qquad \text{and} \qquad \Delta \to 2\cos(\sqrt{\eta}\bar{\sigma}_n) + \sqrt{\eta}\bar{\sigma}_n \sin(\sqrt{\eta}\bar{\sigma}_n), \tag{35}$$

while for the limiting point Ω_5 these limits are modified to

$$\begin{aligned} &v \to -\frac{\sigma^4}{3k}\cos(\sqrt{\eta}\bar{\sigma}_n), \\ &\Delta \to 2\cos(\sqrt{\eta}\bar{\sigma}_n) + \sqrt{\eta}\bar{\sigma}_n \sin(\sqrt{\eta}\bar{\sigma}_n) - \mathrm{i}\frac{\sigma^4}{3k}\cos(\sqrt{\eta}\bar{\sigma}_n). \end{aligned} \tag{36}$$

Using next asymptotics (32), (35) and (36) as well as equation (33), in the limit $l \to 0$ we find from equation (14) the following values for the reflection (R) and transmission (T) coefficients:

$$\bar{R}_n \doteq R|_{\sigma=\bar{\sigma}_n} = -\frac{\eta\bar{\sigma}_n^2}{2+\eta\bar{\sigma}_n^2} \qquad \text{and} \qquad \bar{T}_n \doteq T|_{\sigma=\bar{\sigma}_n} = (-1)^n \frac{\sqrt{1+\eta\bar{\sigma}_n^2}}{1+\eta\bar{\sigma}_n^2/2} \tag{37}$$

for the line Ω_4, and

$$\bar{R}_n = -\frac{\eta\bar{\sigma}_n^2 - \mathrm{i}\bar{\sigma}_n^4/3k}{2+\eta\bar{\sigma}_n^2 - \mathrm{i}\bar{\sigma}_n^4/3k} \qquad \text{and} \qquad \bar{T}_n = (-1)^n \frac{\sqrt{1+\eta\bar{\sigma}_n^2}}{1+\eta\bar{\sigma}_n^2/2 - \mathrm{i}\bar{\sigma}_n^4/6k} \tag{38}$$

for the point Ω_5.

Thus, the point interactions which correspond to the line L_2 are fully non-transparent, except for those values of the parameter σ which satisfy equation (33). For this countable set of points lying on the half-line $0 < \sigma < \infty$ or $0 < \lambda < \infty$ and depending on parameter η, the transmission through the δ'-potential is non-zero ($0 < \bar{T}_n < 1$) for all positive integers n.

Similarly, at the point $\Omega_6 \doteq \{\mu = \nu = 2\}$, from asymptotics (22) one obtains both non-zero limits: $pl \to \sigma$ and $qr \to \sqrt{\eta}\sigma$ as $l \to 0$, so that

$$u \to (\eta^{1/2} + \eta^{-1/2})\sinh\sigma \sin(\eta^{1/2}\sigma), \tag{39}$$

while $v \to \infty$ and $\Delta \to \infty$, except for those values of the constant σ which satisfy the equation

$$\tan(\sqrt{\eta}\sigma) = \sqrt{\eta}\tanh\sigma. \tag{40}$$

This equation also gives a discrete set of positive solutions which we denote by $\{\sigma_n\}_{n=1}^{\infty}$. Again, each σ_n can be found in the same interval (34). At these values, besides asymptotics (39) we have the limits

$$\begin{aligned} &v \to 0, \\ &\Delta \to 2\cosh\sigma_n \cos(\eta^{1/2}\sigma_n) + (\eta^{1/2} - \eta^{-1/2})\sinh\sigma_n \sin(\eta^{1/2}\sigma_n) \end{aligned} \tag{41}$$

as $l \to 0$. Using next asymptotics (39) and (41) as well as equation (40), we find from equation (14) the scattering coefficients:

$$\begin{aligned} &R_n \doteq R|_{\sigma=\sigma_n} = -\frac{(1+\eta)\tanh^2\sigma_n}{2+(\eta-1)\tanh^2\sigma_n}, \\ &T_n \doteq T|_{\sigma=\sigma_n} = (-1)^n \frac{\sqrt{1+\eta\tanh^2\sigma_n}}{\cosh\sigma_n\left(1+\frac{\eta-1}{2}\tanh^2\sigma_n\right)}. \end{aligned} \tag{42}$$

Thus, the point interactions which correspond to the point Ω_6 are fully non-transparent, except for those values of the parameter σ which satisfy equation (40), where the transmission is non-zero ($0 < T_n < 1$). Again, equation (40) acts as a constraint on the coupling constant λ. The solution of this equation also represents a discrete subset in the σ- or λ-space.

In the particular case when $\eta = 1$ (the antisymmetric case) condition (40) takes the simpler form given by equation (10) and found earlier in [19]. Note that only the last case with $\mu = \nu = 2$ corresponds to unrenormalized δ'-interaction (8), because in this case the sequence of squeezing rectangles (18) with $a = 2/(\eta + 1)$ and $b = 2/\eta(\eta + 1)$ weakly converges to the first derivative of Dirac's delta function as $l \to 0$.

4. Boundary conditions on wavefunctions in the zero-range limit

In this section we consider the boundary conditions on the wavefunction $\psi(x)$ at the point $x = 0$ when $l \to 0$. Using the finite-range equations (12)–(17), we find the following expressions (with accuracy to an arbitrary constant) for the left (at $x = -l$) and right (at $x = r$) boundary values of the wavefunction $\psi(x)$:

$$\begin{aligned}\psi(-l) &= \frac{2}{\Delta}\left\{\left[\cosh(pl) - \frac{\mathrm{i}k}{p}\sinh(pl)\right]\cos(qr)\right.\\ &\qquad \left. - \left[\frac{q}{p}\sinh(pl) + \frac{\mathrm{i}k}{q}\cosh(pl)\right]\sin(qr)\right\}\mathrm{e}^{-\mathrm{i}kl},\\ \psi(r) &= \frac{2}{\Delta}\,\mathrm{e}^{-\mathrm{i}kl},\end{aligned} \tag{43}$$

$$\begin{aligned}\psi'(-l) &= \frac{2}{\Delta}\left\{[-p\sinh(pl) + \mathrm{i}k\cosh(pl)]\cos(qr)\right.\\ &\qquad \left. + \left[q\cosh(pl) + \frac{\mathrm{i}kp}{q}\sinh(pl)\right]\sin(qr)\right\}\mathrm{e}^{-\mathrm{i}kl},\\ \psi'(r) &= \frac{2\mathrm{i}k}{\Delta}\,\mathrm{e}^{-\mathrm{i}kl}.\end{aligned} \tag{44}$$

First we note that for the region of full transmission Ω_f in boundary conditions (2) we have $\alpha = \rho = 1$ and $\beta = \gamma = 0$. As regards the boundary conditions on the line L_1, from equations (43) and (44) in the limit $l, r \to 0$ we obtain with accuracy to a constant the following values at $x = \pm 0$:

$$\psi(-0) = \psi(+0) = 1, \qquad \psi'(-0) = \mathrm{i}k - g, \qquad \psi'(+0) = \mathrm{i}k \tag{45}$$

where the constant g is given by equation (29). Equations (45) mean that boundary conditions (2) with values (6) are satisfied, i.e., the renormalized interaction $\lambda\delta'(x)$ on the line L_1 in fact describes the δ-well with constant (29).

Consider now the line Ω_4 and the point Ω_6. For these sets in the zero-range limit ($l, r \to 0$) from equations (43) and (44) we find the following boundary values at $x = \pm 0$ (with accuracy to a constant):

$$\psi(-0) = \rho, \qquad \psi(+0) = 1, \qquad \psi'(-0) = \mathrm{i}k/\rho, \qquad \psi'(+0) = \mathrm{i}k, \tag{46}$$

with

$$\rho = \cos(\sqrt{\eta}\bar{\sigma}_n) \quad \text{for} \quad \Omega_4 \qquad \text{and} \qquad \rho = \frac{\cos(\sqrt{\eta}\sigma_n)}{\cosh\sigma_n} \quad \text{for} \quad \Omega_6. \tag{47}$$

Therefore boundary conditions (2) with constraint (3) are satisfied if $\alpha = \rho^{-1}$ and $\beta = \gamma = 0$, so that the relations

$$\frac{\psi(-0)}{\psi(+0)} = \frac{\psi'(+0)}{\psi'(-0)} = \rho \qquad (48)$$

are valid.

As regards the point Ω_5, here the boundary values (46) are modified to

$$\psi(-0) = \rho, \qquad \psi(+0) = 1, \qquad \psi'(-0) = \mathrm{i}k/\rho - g, \qquad \psi'(+0) = \mathrm{i}k, \qquad (49)$$

where ρ is the same as for Ω_4 (see (47)) and

$$g = -\left(\bar{\sigma}_n^4/3\right)\cos(\sqrt{\eta}\bar{\sigma}_n). \qquad (50)$$

Thus, on the line L_2 and the point Ω_6 both the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ appear to be discontinuous at the singularity point $x = 0$.

Finally, for those points where the fully non-transparent regime takes place we have the limits

$$\psi(-0) = \psi(+0) = 0, \qquad \psi'(-0) = 2\mathrm{i}k, \; \psi'(+0) = 0. \qquad (51)$$

This type of boundary conditions describes the situation when the subsystems in the positive and negative half-lines are totally separated.

5. Conclusions

In this paper we have studied the different point interactions of the dipole type constructed from barrier-well rectangles in the zero-range limit, when both the barrier and the well are squeezed to zero width. The construction of the rectangle sequence involves explicitly three arbitrary positive parameters. One of these parameters (η) controls the width ratio of the barrier and the well, so that when $\eta = 1$ the barrier-well potential becomes an antisymmetric function. The others two describe the rate of increasing the barrier height (μ) and the well depth (ν). As a result, the scattering properties of the whole family of point dipole interactions have been analysed in detail and the corresponding diagram of the existence of interactions with non-trivial transparency properties has been drawn on the $\{\mu, \nu\}$-plane (see figure 1). Summarizing these results, one can conclude that the point dipole interactions are divided into four types as follows: (i) fully transparent, (ii) non-transparent when the subsystems in the positive and negative half-lines are decoupled, (iii) partially transparent acting effectively as a δ-interaction and locating on the line L_1 which separates the fully transparent and non-transparent interactions and (iv) resonances with a discrete subset in the coupling constant space lying (see the line L_2 and the point Ω_6) in the non-transparent region which are also partially transparent but the boundary conditions indeed correspond to a δ'-interaction. Note that only the point set Ω_6 corresponds to the real (unrenormalized) potential $\delta'(x)$, while the other sets are associated with the whole family of different renormalized versions of the δ'-interaction.

The presence of the asymmetry parameter η allows us for any sufficiently large coupling constant λ to construct a regularized sequence of barrier-well rectangles of the type (18) which in the zero-range limit weakly converges to a δ'-potential with non-zero transparency. Indeed, rewriting equation (40) for the potential $\lambda\delta'(x)$ in terms of η as

$$\tan\sqrt{\frac{2\eta}{\eta+1}\lambda} = \sqrt{\eta}\tanh\sqrt{\frac{2}{\eta+1}\lambda}, \qquad (52)$$

one can conclude that for a coupling constant λ exceeding the value $g = 9\pi^2/8$, there exists at least one value of the parameter η which satisfies this equation. This means that for any λ an appropriate sequence of squeezed rectangles can be constructed such that the potential $\lambda\delta'(x)$ is transparent. Therefore the discrete subset of the λ-space being the roots of equation (10) is not a 'distinguished' set which only provides the transparency of this singular potential as could be concluded from the results of recent paper [19]. In fact, the non-zero transparency can be achieved for *any* sufficiently large value of the parameter λ. To this end, the sequence of barrier-well rectangles should be constructed as described at the end of section 3 by using the parameter η.

Thus, in this paper we have studied a more general class of δ'-like point interactions generalizing Šeba's results [15], i.e., instead of a single point Ω_3 we have obtained the whole line L_1 of non-trivial point interactions. We have also generalized the recent results [19] modifying equation (10) to one-parameter equation (40) for a discrete set of values for the coupling constant λ at which the transparency is non-zero. A similar equation, namely (33), has been derived for renormalized δ'-like interactions, also giving a one-parameter discrete subset in the λ-space. It is important to emphasize that the scattering properties of the point dipole depend on the way in which the zero-range limit is accomplished. Thus, for instance, using the right-hand repeated limit in inequality (31), we have obtained the different value of discontinuity of $\psi'(x)$ at $x = 0$ compared with that obtained previously by Šeba [15]. Using this repeated limit, we have also obtained the whole family of δ'-interactions which cannot be obtained by performing the procedure through the left-hand repeated limit in (31). Moreover, while using the right-hand repeated limit, we have shown that the result depends on the form of regularizing sequence, in contrast to δ-interaction (5) for which the solution is unique.

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